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Inventory Management and Markup Pricing in the Presence of Price Fluctuations

Bestandsmanagement und Preisgestaltung bei Vorliegen von Preisschwankungen

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Abstract: We consider a single-item, multi-period inventory model where for a retailer purchase and sales prices fluctuate randomly. We assume that a time-homogeneous and Markovian stochastic market price process represents purchase prices for the retailer and determines the sales prices through a markup rate. Customers arrive according to a doubly-stochastic Poisson process where stochastic arrival rates are affected by the sales prices. Upon observing the inventory level and market price, the retailer decides on an order quantity in each period to maximise the expected profits. We provide conditions on the price process that ensure the concavity of the profit functions and show that a price-dependent base-stock policy is optimal. In a numerical study, we implement a simulation model to estimate the expected profit functions and to analyse the effect of price volatility on the optimal expected profits and base-stock levels. We observe that a more volatile price process leads to lower optimal expected profits.

1 Introduction

Price uncertainties are among the most critical challenges that retailers and manufacturers have to face. Companies whose operations require procuring from commodity markets are exposed to commodity price fluctuations which in fact experienced sharp movements in the near past. Successful inventory management is an effective approach to mitigate risks due to input or even sales prices. Besides its importance in managing the usual trade-off between holding, shortage and purchase costs, it can create additional value in fluctuating price environments by adjusting the order sizes in response to price. Even more value can be created by integrating pricing decisions with inventory planning as pricing is one of the most effective tools a firm has to control the demand (Chan et al. 2004).

Despite their practical significance, price fluctuations are usually not directly addressed in inventory management literature. Classical inventory models usually take purchase and selling prices as constants. It is clear that for some products, input

prices are volatile. Moreover, it is common that selling prices in some industries may be difficult to predict as well. For instance, a wholesaler that sells in a different currency will bear an exchange rate risk in sales prices. Some industries such as apparel and high technology face the problem of variable selling price due to rapid product substitution and short life cycles. As another example, one can think of jewellery store managers who buy and sell products that are made of precious metals such as gold, silver and precious gemstones such as diamond. Sales prices of these products will be randomly varying over time and customers arriving at different times may be charged different prices.

In this paper, we investigate the inventory management problem of a retailer that sells a commodity-based product whose inherent value is changing according to a stochastic price process. In a multi-period, single-item, periodic review setting, we explicitly model a stochastic market price process which determines both purchase and sales prices and also stochastically describes the customer demand. To incorporate the effect of stochastic sales prices, we model the customer arrival process which is modulated by the market price process such that times of customer arrivals become important in determination of sales prices. In a numerical study, we use Monte Carlo simulation to compute expected profits by first generating random price paths from a continuous price process and then the conditional customer arrival process. We generate insights on the effect of price volatilities on retailer's optimal expected profits and inventory decisions.

The rest of the paper is organised as follows. In Section 2, we briefly review the relevant literature. In Section 3, we present the inventory control model involving price fluctuations. In Section 4, we use simulation to conduct a sensitivity analysis on the optimal values of the models developed before. Lastly, in Section 5, we give our concluding remarks and further research ideas.

2 Literature

Several researchers studied the effect of volatile purchase prices on inventory control problems. Kalymon (1971) studies a multi-period inventory model by incorporating the random purchase prices which are determined by a Markov process. He proves that a state-dependent (s, S) policy is optimal. Berling and Martinez de Albeniz (2011) investigate a Poisson demand system where purchase price is a Markov process to see the effect of price evolution on the optimal policy and its parameters. There is also a wide range of models that consider the use of volatile spot markets where the firms can buy and sell. Goel and Gutierrez (2006) consider a multi-period stochastic inventory model where a firm may purchase from both spot and future markets. Haksoz and Seshadri (2007) review existing models that incorporate spot market procurements with volatile prices in several supply chain operations. Secomandi (2010) studies the warehouse problem of a merchant that involves in commodity-trading activities. He assumes that at each period, the spot price of the commodity evolves as a Markov process. In the presence of both space and buy-sell limits, he shows that optimal policy is characterised by two-stage price dependent base-stock targets.

Despite numerous papers that consider stochastic input prices, few papers investigate the effect of changing sales prices which are mostly due to inflation or a deterministic continuous price decrease. Erel (1992) demonstrates the sensitivity of

the basic model to continuous price changes and concludes that relationship between price change rate and holding cost rate is highly important. Hariga (1995) studies EOQ models with linearly increasing price and demand. Khouja and Park (1993) consider EOQ models with decreasing purchase price motivated by the statistical finding that in high technology industry, component prices decline at a rate of 1% per week. Banerjee and Meitei (2009) consider a single period stochastic demand inventory model with random lead time and continuously decreasing selling price.

The model presented in this paper differs from existing models in the sense that it incorporates the effect of random sales prices by explicitly modelling a continuous-state stochastic price process. It also relates random purchase and sales prices through a proportional retail markup unlike classical models that take sales price as constants or a random variable unrelated to the purchase costs. This case is applicable to situations where a firm is selling in a foreign currency or selling a commodity-based item such that any price fluctuations in the material cost also pass to customers. In addition, besides using random variables for random demands to be realised at the end of sales periods, we model the customer demand also as a stochastic process that depends on the prevailing stochastic prices. The next section presents the details of our model.

3 The Model

We consider a multi-period and periodic-review inventory model with fluctuating prices. We assume that there are M periods whose lengths are equal to T units of time where at the beginning of each sales period the retailer places an order. For the model in consideration, we assume the existence of a stochastic market price that follows a continuous-time and time-homogeneous Markov process $P = \{P_t;\ t \geq 0\}$ with state space $\mathbb{R}_+ = [0, \infty)$. We assume that these random market prices actually represent the purchase prices of the retailer in the analysis, i.e., the value of the market price at review times (i.e., P_T, P_{2T}, ..) are the purchase prices for the retailer. Unlike most of the inventory models that model the customer demand as a random variable to be realised at the end of each review period, we assume that there is a customer arrival process and it is modulated by the market price process that we consider. More specifically, we assume that the unit customer demand process is a modulated Poisson process where stochastic arrival rate at time t is $\Lambda_t = \lambda(P_t)$ and $\lambda(.)$ is a nonnegative function of the random prices. Customers arrive according to this process and at each arrival they demand one unit of the item. We here note that all the following analysis holds if each customer demands a random amount of the item provided that they are independent and identically distributed. In our setting, since arrival rate is a function of stochastic price, we have a stochastic arrival rate process $\Lambda = \{\Lambda_t;\ t \geq 0\}$ which modulates the customer arrival process. These types of models are referred as doubly stochastic Poisson processes introduced by Cox (1955) or shortly, Cox processes. If Λ is a deterministic function rather than a stochastic process, we have a nonhomogeneous Poisson process. Since we assume that P is a Markov process, Λ is also Markovian. We denote the customer purchase process by $N = \{N_t;\ t \geq 0\}$ where N_t denotes the number of sales by time t and $N_0 = 0$. The arrival times of the customers form a random sequence $S = \{T_n;\ n \geq 1\}$ where S and N are related as $\{T_n \leq t\} = \{N_t \geq n\}$.

Sales prices are assumed to be proportional to the market price with a fixed markup ratio of $\alpha \geq 1$. In other words, if a customer arrives at time t, the sales price is αP_t where P_t is the current random market price. This is appropriate for products that have constantly changing underlying commodity prices like diamond rings, gold bracelets etc. For such products, it is common practice to set a proportional markup rather than an incremental markup. We assume that α is an exogenous economic parameter to the retailer which is plausible for environments with high competition. The type of the product that the retailer in question sells is also relevant. For a gold retailer, for example, setting a retail markup freely will be very difficult for gold coins which are non-differentiated products and easily be substituted by customers at other retailers. On the other hand, for some exclusive products like designer bracelets, necklaces, etc., the retailer may have more freedom in setting his markup since an arriving customer may not find exactly the same product elsewhere. Although we assume a fixed markup rate α in the analysis of this section, in the numerical analysis of Section 4 we analyse the case where the retailer also sets the markup together with the inventory decision.

At the beginning of each period, the decision maker observes the current price and inventory level to make an ordering decision. There is no lead time and the entire order is received immediately at the beginning of each period. Customers that arrive in the case of shortage are assumed to be lost. We assume that for each period, unit inventory holding is $h \geq 0$ and unit lost-sale cost is $b \geq 0$. Moreover an interest rate parameter $r \geq 0$ is used for discounting the revenues. The following analysis also holds if these parameters are functions of initial price at each period.

The objective is to maximise the immediate and subsequent discounted profits. Since we assume that the unit demands arrive according to a Cox process which is modulated by a Markovian price process and the length of intervals are the same, the probability distribution of number of sales in any interval only depends on the initial price at the beginning of that period. In other words they are conditionally independent. We denote the total demand in a period by N_T, whose distribution depends on the initial market price P_0. We also do not put any restriction on the price process except the Markov property. Since we assume that each demand is of size one, the state space for inventory level is nonnegative integers.

We can write the total expected revenue during a period as a function of initial price p and order-up-to decision y as

$$r(y;p) = E\left[\sum_{n=1}^{\min(N_T,y)} e^{-rT_n}\alpha P_{T_n} \mid P_0 = p\right] = \sum_{n=1}^{y} E\left[e^{-rT_n}\alpha P_{T_n} 1_{\{T_n \leq T\}} \mid P_0 = p\right] \qquad (1)$$

where N_T denotes the number of arrivals during the period and T_n is the time of nth arrival since the beginning of the period. The summation inside the expectation in 1 is the total discounted revenue in which the item is sold, if available, at the current sales price αP_{T_n} upon arrival of the nth customer since the beginning of the period. Summation is performed until the last sale and every time revenue is collected, it is discounted to the beginning of the period. The expectation in 1 is taken with respect to the random components; number of arrivals, the time of arrivals and the market price. A similar approach appears in Grubbstorm (2007) who considers a single-period problem where demand is modelled as a compound renewal process. He

assumes that sales price is constant and customers that arrive according to a renewal process demand a random amount of the product. There is no fixed sales period and items are sold until the inventory is depleted. Although in our model we sum all individual revenues from each arriving customer, our model construction is different in the sense that we have a finite selling season and sales price is a stochastic process that also modulates the customer arrival process.

The total expected discounted one-period profit can be written similarly as

$$g(y;p) = -py + r(y;p) - E\left[b(N_T - y)^+ + h(y - N_T)^+ \mid P_0 = p\right] \tag{2}$$

The first term in 2 is the total purchase cost for y units at the initial price p and the last term is the one-period expected lost-sale and inventory holding costs. Note that the one-period expected profit is independent of the period. This is due to the fact that conditional random prices $P_{kT+T_n} \mid P_{kT}$ and $P_{T_n} \mid P_0$ have the same distribution for any period k since stochastic market price process is assumed to be Markovian and time-homogenous.

We use dynamic programming to solve the inventory problem to optimality. We define the value function $V_k(x, p)$ as the maximum total expected discounted profit for periods from k to M if the initial inventory is x and market price is p. We write the dynamic programming equation as

$$V_k(x,p) = \max_{y \geq x}\left\{g(y;p) + \gamma\Psi_k(y;p)\right\} + px \tag{3}$$

where

$$\Psi_k(y;p) = E\left[V_{k+1}\left((y - N_T)^+, P_T\right) \mid P_0 = p\right] \tag{4}$$

denotes the total expected future profits from period $k + 1$ to M. Note that the discount factor for a sales period is $\gamma = e^{-rT}$. Since we do not allow backorders, the inventory level can not be negative in the next period. It should be zero if the demand turns out to be more than the total inventory in the current period. We assume that the salvage price is zero. Therefore, the terminal value function is

$$V_{M+1}(x,p) = 0\,. \tag{5}$$

Lemma 1: A necessary condition for $g(y;p)$ to be concave in y is

$$E\left[1_{\{T_n \leq T\}}\left(e^{-rT_n}\alpha P_{T_n} + b + h\right) \mid P_0 = p\right] \tag{6}$$

is decreasing in n. Moreover, if the expected discounted prices are nonincreasing in time, the condition given in 6 is always satisfied.

Proof: Note that forward difference of $g(y;p)$ can be written as

$$\begin{aligned} g(y+1;p) - g(y;p) &= -p + E\left[e^{-rT_n}\alpha P_{T_{y+1}} 1_{\{T_{y+1} \leq T\}} \mid P_0 = p\right] \\ &\quad + bP\{N_T \geq y+1 \mid P_0 = p\} - hP\{N_T \leq y \mid P_0 = p\} \\ &= -(p+h) + E\left[1_{\{T_{y+1} \leq T\}}\left(e^{-rT_n}\alpha P_{T_{y+1}} + b + h\right) \mid P_0 = p\right] \end{aligned} \tag{7}$$

Note that under the condition given in 6, the forward difference of the expected one-period profit function is decreasing in y which ensures that it is integer concave. □

Theorem 1: Assume that the necessary condition is satisfied. Then $V_k(x, p)$ is concave in x for every p and a base-stock policy is optimal, i.e., there exists a base-stock level $S_k(p)$ for each period k such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to $S_k(p)$. Otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period k is given by

$$S_k(p) = \inf\left\{ \begin{array}{l} y \geq 0 : P\{N_T \leq y \mid P_0 = p\} \\ \qquad \geq \dfrac{-p + b + E\left[1_{\{T_y \leq T\}} e^{-rT_y} \alpha P_{T_y} \mid P_0 = p\right] + \gamma \Delta \Psi_k(y; p)}{b + h} \end{array} \right\} \tag{8}$$

Proof: We prove the result by induction. First note that terminal value function is trivially concave. Assuming that that $V_{k+1}(x, p)$ is concave for any $k < M$. This makes $\Psi_k(y; p)$ concave. By lemma 1, since $g(y; p)$ is concave in y, $g(y; p) + \gamma\Psi_k(y; p)$ is concave in y which also makes $V_k(x, p)$ concave in x. This suggests that a price-dependent base-stock policy is optimal. □

4 Numerical Analysis

In this section, we conduct a sensitivity analysis on the model presented in Section 3 and aim to investigate the effect of price volatility on the optimal profits and optimal base-stock levels. We focus on the single period model and through a simulation study, we generate insights on the sensitivity of the optimal profits to price volatility by comparing them for various volatility levels.

In our analysis, we assume that the sales period is $T = 1$ unit of time and market prices follow a geometric Brownian motion process with drift μ and volatility σ. We also assume that the arrival process is a doubly-stochastic Poisson process with the following piecewise linear rate function

$$\Lambda_t = \left(A - B\alpha P_t\right)^+. \tag{9}$$

A direct analytical approach to the above model is challenging. Therefore, we employ a simulation approach to estimate the performance measures. To generate the market prices, we use $n = 100$ equally-spaced discretisation of the interval [0,1] to simulate a random path for a Wiener process $W = \{W_t; 0 \leq t \leq T\}$ where it is well-known that $W_{tk} - W_{tk-1} \sim$ Normal $(0, 1/n)$. We then use these Wiener realisations in the analytical Ito' solution for the geometric Brownian motion which is

$$P_t = P_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}. \tag{10}$$

(Baxter and Rennie 1996). This way we generate a sample path for a geometric Brownian motion with drift μ and volatility σ. Then, conditional on the price path, we generate a nonhomogeneous Poisson process stream for the customer arrivals using the thinning algorithm of Lewis and Shedler (1979). According to the thinning algorithm, for each price path, we set $\lambda_{max} = A$, and generate a Poisson process with rate λ_{max}. At each arrival time, we generate an independent U = Uniform (0, 1) random variable and accept the arrival if $U \leq \Lambda_U / \lambda max$ where

$\Lambda_U = (A - B\alpha P_U)^+$. This way, we generate a nonhomogeneous Poisson process for each realised price path.

We take the initial price $P_0 = 20$ and assume that h = b = r = 0 for simplicity. To investigate the effect of price volatility, we take the drift of the price process $\mu = 0$. This makes the price process in 10 a martingale, i.e., the theoretical expectation is $E[P_t] = P_0$ which means that the process remains the same in expectation. Although increasing the value of price volatility σ does not change the theoretical expected future values of prices, the variability increases. We use 10000 replications to estimate the expected profit function

$$g(y, \alpha; p) = -py + E\left[\sum_{n=1}^{\min(N_T, y)} \alpha P_{T_n} \mid P_0 = p\right]. \tag{11}$$

Note that 11 is the same as 2 when h = b = r = 0 and in addition, the markup α is also a decision variable. We use volatility values of [0, 0.02, .. , 0.8] for the analysis and at first we fix the markup $\alpha = 1.8$. As seen in Figure 1, although optimal base-stock levels that maximise 11 exhibit nonmonotone behaviour, they have a tendency to decrease as price volatility increases.

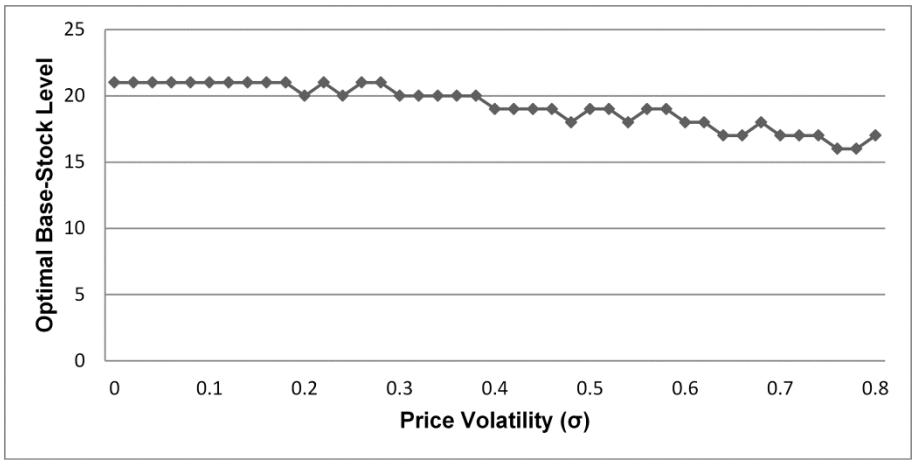

Figure 1: *Sensitivity of the optimal base-stock level to price volatility*

The bottom curve in Figure 2, on the other hand, shows the optimal expected profits for the same price volatility levels when $\alpha = 1.8$ is used as the sales markup. The top curve titled g(y*, α*) shows the optimal expected profits when the expected profit function in 11 is optimised over order quantity y and markup α. We observe that whether the markup is exogenous to the retailer or not, the optimal expected profits decrease as prices get more volatile while their expected future values remain the same. This suggests that price volatility is undesirable for a retailer that faces the risk of random sales price although it has the power of adjusting the order size and setting the retail markup.

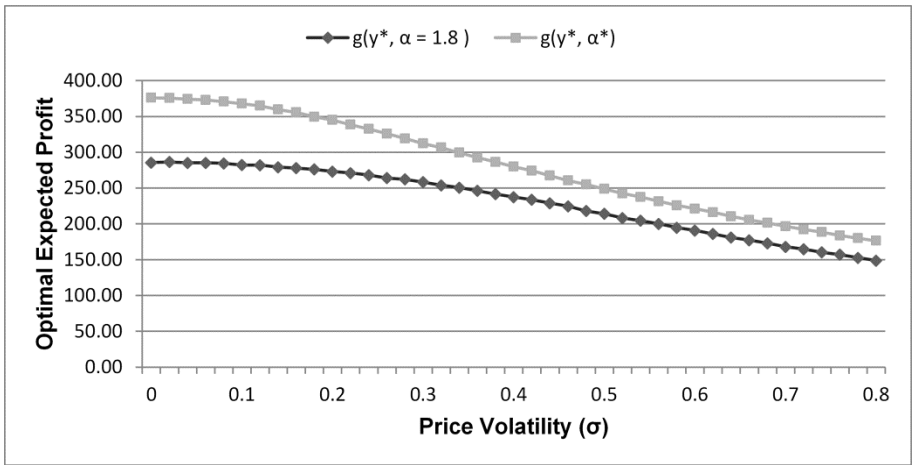

Figure 2: *Sensitivity of the optimal expected profits to price volatility*

5 Conclusion

This paper considers a multi-period inventory management problem of a retailer that faces both stochastic input and sales prices. From a modelling point of view, we assume that inherent value of the product in question is fluctuating due to exchange rates or commodity content. We refer to these fluctuating prices as the market price and assume that they follow a time-homogeneous Markov process. We also model the customer arrival process as a doubly stochastic Poisson process that is governed by a stochastic intensity process where the stochastic intensity at any time is determined by the random market price and the sales markup. We model this inventory management problem using dynamic programming with states being the current inventory level and market price at the review periods. We show that a price-dependent base-stock policy is optimal under a condition on the price process. Moreover, it is sufficient to have non-increasing expected discounted market prices to ensure the concavity of the expected profit functions.

In the last part, we conduct a sensitivity analysis to observe the effect of volatile prices on the optimal decisions and optimal profits. To compute the value of the expected profit functions, we use Monte Carlo simulation as it is rather difficult and time-consuming to compute it analytically for continuous-time, continuous-state price processes. Using the fact that a doubly stochastic Poisson process reduces to a nonhomogeneous Poisson process conditioned on the stochastic arrival rates, we first generate market prices at each replication as they define the arrival rates. We assume that market prices follow a geometric Brownian motion having no drift. For a fixed volatility level, we generate the prices at equidistant time points in the sales season. Then for each realised price path, we generate a nonhomogeneous Poisson stream using a thinning algorithm and estimate the single-period profit function. We observe that as the prices get more volatile, optimal expected profits decrease when the sales markup is a decision variable or not.

References

Banerjee, S.; Meitei, N.: Effect of declining selling price: Profit analysis for a single period inventory model with stochastic demand and lead time. Journal of the Operational Research Society 61 (2009) 4, pp. 696-704.

Baxter, M; Rennie, A.: Financial Calculus An introduction to derivative pricing. Cambridge: Cambridge University Press 1996.

Berling, P; Martínez-de Albéniz, V.: Optimal inventory policies when purchase price and demand are stochastic. Operations Research 59 (2011) 1), pp.109-124.

Chan, L. M.; Shen, Z. M.; Simchi-Levi, D.; Swann. J. L.: Coordination of pricing and inventory decisions: a survey and classification. In: Simchi-Levi, D.; Wu, S.D.; Shen, Z.-J. (eds): Handbook of Quantitative Supply Chain Analysis. New York: Springer Science+Business Media 2004, pp. 335–392.

Cox, D. R.: Some statistical methods connected with series of events. Journal of the Royal Statistical Society. Series B (Methodological) 17 (1955) 2, pp. 129-164.

Erel, E.: The effect of continuous price change in the EOQ. Omega, 20 (1992) 4, pp. 523-527.

Goel, A.; Gutierrez, G.: Integrating commodity markets in the optimal procurement policies of a stochastic inventory system. Available at SSRN 930486, 2006.

Grubbström, R.: The newsboy problem when customer demand is a compound renewal process. European Journal of Operational Research 203 (2010) 1, pp. 134-142.

Haksöz, Ç; Seshadri, S.: Supply chain operations in the presence of a spot market: A review with discussion. Journal of the Operational Research Society 58 (2007) 11, pp. 1412-1429.

Hariga, M.: Effects of inflation and time-value of money on an inventory model with time-dependent demand rate and shortages. European Journal of Operational Research 81 (1995) 3, pp. 512-520.

Kalymon, B.: Stochastic prices in a single item inventory purchasing model. Operations Research 19 (1971) 6, pp. 1434-1458.

Khouja, M; Park, S.: Optimal lot sizing under continuous price decrease. Omega 31(2003) 6, pp. 539-545.

Lewis, P.A. W.; Shedler, G.A.: Simulation of Nonhomogeneous Poisson Processes by Thinning. Naval Research Logistics Quarterly 26 (1979) 3, pp. 403-414.

Secomandi, N.: Optimal commodity trading with a capacitated storage asset. Management Science 56(2010) 3, pp. 449-467.